

Weighted sampling, Maximum Likelihood and minimum divergence estimators

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July 30, 2012

Abstract

This paper explores Maximum Likelihood in parametric models in the context of Sanov type Large Deviation Probabilities. MLE in parametric models under weighted sampling is shown to be associated with the minimization of a specific divergence criterion defined with respect to the distribution of the weights. Some properties of the resulting inferential procedure are presented; Bahadur efficiency of tests are also considered in this context.

1 Motivation and context

This paper explores Maximum Likelihood paradigm in the context of sampling. It mainly quotes that inference criterion is strongly connected with the sampling scheme generating the data. Under a given model, when i.i.d. sampling is considered and some standard regularity is assumed, then the Maximum Likelihood principle loosely states that conditionally upon the observed data, resampling under the same i.i.d. scheme should resemble closely to the initial sample only when the resampling distribution is close to the initial unknown one.

Keeping the same definition it appears that under other sampling schemes, the Maximum Likelihood Principle yields a wide range of statistical procedures. Those have in common with the classical simple i.i.d. sampling case that they can be embedded in a natural class of methods based on minimization of ϕ -divergences between the empirical measure of the data and the model. In the classical i.i.d. case the divergence is the Kullback-Leibler one, which yields the standard form of the Likelihood function. In the case of the weighted bootstrap, the divergence to be optimized is directly related to the distribution of the weights.

This paper discusses the choice of an inference criterion in parametric setting. We consider a wide range of commonly used statistical criteria, namely all those induced

by the so-called power divergence, including therefore Maximum Likelihood, Kullback-Leibler, Chi-square, Hellinger distance, etc. The steps of the discussion are as follows.

We first insert Maximum Likelihood paradigm at the center of the scene, putting forwards its strong connection with large deviation probabilities for the empirical measure. The argument can be sketched as follows: for any putative θ in the parameter set, consider n virtual simulated r.v.'s $X_{i,\theta}$ with corresponding empirical measure $P_{n,\theta}$. Evaluate the probability that $P_{n,\theta}$ is close to P_n , conditionally on P_n , the empirical measure pertaining to the observed data; such statement is referred to as a conditional Sanov theorem, and for any θ this probability is governed by the Kullback-Leibler distance between P_θ and P_{θ_T} where θ_T stands for the true value of the parameter. Estimate this probability for any θ , obviously based on the observed data. Optimize in θ ; this provides the MLE, as shown in the two cases of the i.i.d. sample scheme; our first example is the case when the observations take values in a finite set, and the second case (infinite case), helps to set the arguments to be put forwards. Introducing MLE's through Large deviations for the empirical measure is in the vein of various recent approaches; see Grendar and Judge [7].

We next consider a generalized sampling scheme inherited from the bootstrap, which we call weighted sampling; it amounts to introduce a family of i.i.d. weights W_1, \dots, W_n with mean and variance 1. The corresponding empirical measure pertaining to the data set x_1, \dots, x_n is just the weighted empirical measure. The MLE is defined through a similar procedure as just evoked. The conditional Sanov Theorem is governed by a divergence criterion which is defined through the distribution of the weights. Hence MLE results in the optimization of a divergence measure between distributions in the model and the weighted empirical measure pertaining to the dataset.

Resulting properties of the estimators are studied.

Optimization of ϕ -divergences between the empirical measure of the data and the model is problematic when the support of the model is not finite. A number of authors have considered so-called dual representation formulas for divergences or, globally, for convex pseudodistances between distributions. We will make use of the one exposed in [3]; see also [1] for an easy derivation.

1.1 Notation

1.1.1 Divergences

The space S is a Polish space endowed with its Borel field $\mathcal{B}(S)$. We consider an identifiable parametric model \mathcal{P}_Θ on $(S, \mathcal{B}(S))$, hence a class of probability distributions P_θ indexed by a subset Θ included in \mathbb{R}^d ; Θ needs not be open. The class of all probability measures on $(S, \mathcal{B}(S))$ is denoted \mathcal{P} and $\mathcal{M}(S)$ designates the class of all finite signed measures on $(S, \mathcal{B}(S))$.

A non negative convex function φ with values in $\overline{\mathbb{R}^+}$ belonging to $C^2(\mathbb{R})$ and satisfying $\varphi(1) = \varphi'(1) = 0$ and $\varphi''(1)$ is a *divergence function*. An important class of such functions

is defined through the power divergence functions

$$\varphi_\gamma(x) := \frac{x^\gamma - \gamma x + \gamma - 1}{\gamma(\gamma - 1)} \quad (1.1)$$

defined for all real $\gamma \neq 0, 1$ with $\varphi_0(x) := -\log x + x - 1$ (the likelihood divergence function) and $\varphi_1(x) := x \log x - x + 1$ (the Kullback-Leibler divergence function). This class is usually referred to as the Cressie-Read family of divergence functions, a custom we will follow, although its origin takes from [12]. When x is such that $\varphi_\gamma(x)$ is undefined by the above definitions, we set $\varphi_\gamma(x) := +\infty$, by which the definition above is satisfied for all φ_γ . It consists in the simplest power-type class of functions (with the limits in $\gamma \rightarrow 0, 1$) which fulfill the definition. The L_1 divergence function $\varphi(x) := |x - 1|$ is not captured by the Cressie-Read family of functions.

Associated with a divergence function φ is the *divergence pseudodistance* between a probability measure and a finite signed measure; see [4].

For P and Q in \mathcal{M} define

$$\begin{aligned} \phi(Q, P) &:= \int \varphi\left(\frac{dQ}{dP}\right) dP \quad \text{whenever } Q \text{ is a.c. w.r.t. } P \\ &:= +\infty \quad \text{otherwise.} \end{aligned}$$

The divergence $\phi(Q, P)$ is best seen as a mapping $Q \rightarrow \phi(Q, P)$ from \mathcal{M} onto $\overline{\mathbb{R}^+}$ for fixed P in \mathcal{M} . Indexing this pseudodistance by γ and using φ_γ as divergence function yields the likelihood divergence $\phi_0(Q, P) := -\int \log\left(\frac{dQ}{dP}\right) dP$, the Kullback-Leibler divergence $\phi_1(Q, P) := \int \log\left(\frac{dQ}{dP}\right) dQ$, the Hellinger divergence $\phi_{1/2}(Q, P) := \frac{1}{2} \int \left(\sqrt{\frac{dQ}{dP}} - 1\right)^2 dP$, the modified χ^2 divergence $\phi_{-1}(Q, P) := \frac{1}{2} \int \left(\frac{dQ}{dP} - 1\right)^2 \left(\frac{dQ}{dP}\right)^{-1} dP$. All these divergences are defined on \mathcal{P} . The χ^2 divergence $\phi_2(Q, P) := \frac{1}{2} \int \left(\frac{dQ}{dP} - 1\right)^2 dP$ is defined on \mathcal{M} . We refer to [3] for the advantage to extend the definition to possibly signed measures in the context of parametric inference for non regular models.

The conjugate divergence function of φ is defined through

$$\tilde{\varphi}(x) := x\varphi\left(\frac{1}{x}\right) \quad (1.2)$$

and the corresponding divergence pseudodistance $\tilde{\phi}(P, Q)$ is

$$\tilde{\phi}(P, Q) := \int \tilde{\varphi}\left(\frac{dP}{dQ}\right) dQ$$

which satisfies

$$\tilde{\phi}(P, Q) = \phi(Q, P)$$

whenever defined, and equals $+\infty$ otherwise. When $\varphi = \varphi_\gamma$ then $\tilde{\varphi} = \varphi_{1-\gamma}$ as follows by substitution. Pairs $(\varphi_\gamma, \varphi_{1-\gamma})$ are therefore *conjugate pairs*. Inside the Cressie-Read family, the Hellinger divergence function is self-conjugate.

In parametric models φ -divergences between two distributions take a simple variational form. It holds, when φ is a differentiable function, and under a commonly met regularity condition, denoted (RC) in [1]

$$\phi(P_\theta, P_{\theta_T}) = \sup_{\alpha \in \mathcal{U}} \int \varphi' \left(\frac{dP_\theta}{dP_\alpha} \right) dP_\theta - \int \varphi^\# \left(\frac{dP_\theta}{dP_\alpha} \right) dP_{\theta_T} \quad (1.3)$$

where $\varphi^\#(x) := x\varphi'(x) - \varphi(x)$. In the above formula, \mathcal{U} designates a subset of Θ containing θ_T such that for any θ, θ' in \mathcal{U} , $\phi(P_\theta, P_{\theta'})$ is finite. This formula holds for any divergence in the Cressie Read family, as considered here.

Denote

$$h(\theta, \alpha, x) := \int \varphi' \left(\frac{dP_\theta}{dP_\alpha} \right) dP_\theta - \varphi^\# \left(\frac{dP_\theta}{dP_\alpha}(x) \right)$$

from which

$$\phi(P_\theta, P_{\theta_T}) := \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha, x) dP_{\theta_T}(x). \quad (1.4)$$

For CR divergences

$$h(\theta, \alpha, x) = \frac{1}{\gamma - 1} \left[\int \left(\frac{dP_\theta}{dP_\alpha} \right)^{\gamma-1} dP_\theta - 1 \right] - \frac{1}{\gamma} \left[\left(\frac{dP_\theta}{dP_\alpha}(x) \right)^\gamma - 1 \right].$$

1.1.2 Weights

For a given real valued random variable W denote

$$M(t) := \log E \exp tW \quad (1.5)$$

its cumulant generating function which we assume to be finite in a non void interval including 0 (this is the so-called Cramer condition). The Fenchel Legendre transform of M is also called the Chernoff function and is defined through

$$\varphi^W(x) = M^*(x) := \sup_t tx - M(t). \quad (1.6)$$

The function $x \rightarrow \varphi^W(x)$ is non negative, is C^2 and convex. We also assume that $EW = 1$ together with $VarW = 1$ which implies $\varphi^W(1) = (\varphi^W)'(1) = 0$ and $(\varphi^W)''(1) = 1$. Hence $\varphi^W(x)$ is a divergence function with corresponding divergence pseudodistance ϕ^W . Associated with φ^W is the conjugate divergence $\widehat{\phi}^W$ with divergence function $\widetilde{\varphi}^W$, which therefore satisfies

$$\phi^W(Q, P) = \widehat{\phi}^W(P, Q).$$

1.1.3 Measure spaces

This paper makes extensive use of Sanov type large deviation results for empirical measures or weighted empirical measures. This requires some definitions and facts.

The vector space $\mathcal{M}(S)$ is endowed with the τ -topology, which is the coarsest making all mappings $Q \rightarrow \int f dQ$ continuous for any $Q \in \mathcal{M}(S)$ and any $f \in B(S)$ which denotes the class of all bounded measurable functions on $(S, \mathcal{B}(S))$. A slightly stronger topology will be used in this paper, the τ_0 topology, introduced in [5], which is the natural setting for our sake. This topology can be described through the following basis of neighborhoods. Consider \mathfrak{P} the class of all partitions of S and for $k \geq 1$ the class \mathfrak{P}_k of all partitions of S into k disjoint sets, $\mathcal{P}_k := (A_1, \dots, A_k)$ where the A_i 's belong to $\mathcal{B}(S)$. For fixed P in \mathcal{M} , for any k , any such partition \mathcal{P}_k in \mathfrak{P}_k and any positive ε define the open neighborhood $U(P, \varepsilon, \mathcal{P}_k)$ through

$$U(P, \varepsilon, \mathcal{P}_k) := \left\{ Q \in \mathcal{M} \text{ such that } \max_{1 \leq i \leq k} |P(A_i) - Q(A_i)| < \varepsilon \text{ and } Q(A_i) = 0 \text{ if } P(A_i) = 0 \right\}.$$

The additional requirement $Q(A_i) = 0$ if $P(A_i) = 0$ in the above definition with respect to the classical definition of the basis of the τ -topology is essential for the derivation of Sanov type theorems. Endowed with the τ_0 -topology, \mathcal{M} is a Hausdorff locally convex vector space.

The following Pinsker type property holds

$$\sup_k \sum_{i=1}^k \varphi \left(\frac{Q(A_i)}{P(A_i)} \right) P(A_i) = \phi(Q, P)$$

see [8].

For any P in \mathcal{M} the mapping $Q \rightarrow \phi(Q, P)$ is lower semi continuous; see [2], Proposition 2.2. Denoting (a, b) the domain of φ whenever

$$\lim_{\substack{x \rightarrow a \\ x > a}} \frac{\varphi(x)}{x} = \lim_{\substack{x \rightarrow b \\ x < b}} \frac{\varphi(x)}{x} = +\infty$$

then for any positive C , the level set $\{Q : \phi(Q, P) \leq C\}$ is τ_0 -compact, making $Q \rightarrow \phi(Q, P)$ a so-called good rate function. Divergence functions φ satisfying this requirement for example are φ_γ with $\gamma > 1$; see [2] for different cases.

1.1.4 Minimum dual divergence estimators

The above formula (1.3) defines a whole range of plug in estimators of $\phi(P_\theta, P_{\theta_T})$ and of θ_T . Let X_1, \dots, X_n denote n i.i.d. r.v's with common didistribution P_{θ_T} . Denoting

$$P_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

the empirical measure pertaining to this sample. The plug in estimator of $\phi(P_\theta, P_{\theta_T})$ is defined through

$$\phi_n(P_\theta, P_{\theta_T}) := \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha, x) dP_n(x)$$

and the family of M-estimators indexed by θ

$$\alpha_n(\theta) := \arg \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha, x) dP_n(x)$$

approximates θ_T . In the above formulas \mathcal{U} is defined after (1.3). See [3] and [13] for asymptotic properties and robustness results.

Since $\phi(P_{\theta_T}, P_{\theta_T}) = 0$ a natural estimator of θ_T which only depends on the choice of the divergence function φ is defined through

$$\begin{aligned} \theta_n &:= \arg \inf_{\theta} \phi_n(P_\theta, P_{\theta_T}) \\ &= \arg \inf_{\theta \in \mathcal{U}} \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha, x) dP_n(x); \end{aligned}$$

see [3] for limit properties.

2 Large deviation and maximum likelihood

2.1 Maximum likelihood under finite supported distributions and simple sampling

Suppose that all probability measures P_θ in \mathcal{P}_Θ share the same finite support $S := \{1, \dots, k\}$. Let X_1, \dots, X_n be a set of n independent random variables with common probability measure P_{θ_T} and consider the Maximum Likelihood estimator of θ_T . A common way to define the ML paradigm is as follows: For any θ consider independent random variables $(X_{1,\theta}, \dots, X_{n,\theta})$ with probability measure P_θ , thus *sampled in the same way as the X_i 's*, but under some alternative θ . Define θ_{ML} as the value of the parameter θ for which the probability that, up to a permutation of the order of the $X_{i,\theta}$'s, the probability that $(X_{1,\theta}, \dots, X_{n,\theta})$ occupies S as does X_1, \dots, X_n is maximal, conditionally on the observed sample X_1, \dots, X_n . In formula, let σ denote a random permutation of the indexes $\{1, 2, \dots, n\}$ and θ_{ML} is defined through

$$\theta_{ML} := \arg \max_{\theta} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}} P_\theta \left((X_{\sigma(1),\theta}, \dots, X_{\sigma(n),\theta}) = (X_1, \dots, X_n) \mid (X_1, \dots, X_n) \right) \quad (2.1)$$

where the summation is extended on all equally probable permutations of $\{1, 2, \dots, n\}$.

Denote

$$P_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

and

$$P_{n,\theta} := \frac{1}{n} \sum_{i=1}^n \delta_{X_{i,\theta}}$$

the empirical measures pertaining respectively to (X_1, \dots, X_n) and $(X_{1,\theta}, \dots, X_{n,\theta})$

An alternative expression for θ_{ML} is

$$\theta_{ML} := \arg \max_{\theta} P_{\theta} (P_{n,\theta} = P_n | P_n). \quad (2.2)$$

An explicit enumeration of the above expression $P_{\theta} (P_{n,\theta} = P_n | P_n)$ involves the quantities

$$n_j := \text{card} \{i : X_i = j\}$$

for $j = 1, \dots, k$ and yields

$$P_{\theta} (P_{n,\theta} = P_n | P_n) = \frac{\prod_{j=1}^k n_j! P_{\theta}(j)^{n_j}}{n!} \quad (2.3)$$

as follows from the classical multinomial distribution. Optimizing on θ in (2.3) yields

$$\begin{aligned} \theta_{ML} &= \arg \max_{\theta} \sum_{j=1}^k \frac{n_j}{n} \log P_{\theta}(j) \\ &= \arg \max_{\theta} \frac{1}{n} \sum_{i=1}^n \log P_{\theta}(X_i). \end{aligned}$$

Consider now the Kullback-Leibler distance between P_{θ} and P_n which is non commutative and defined through

$$\begin{aligned} KL(P_n, P_{\theta}) &:= \sum_{j=1}^k \varphi\left(\frac{n_j/n}{P_{\theta}(j)}\right) P_{\theta}(j) \\ &= \sum_{j=1}^k (n_j/n) \log \frac{n_j/n}{P_{\theta}(j)} \end{aligned} \quad (2.4)$$

where

$$\varphi(x) := x \log x - x + 1 \quad (2.5)$$

which is the Kullback-Leibler divergence function. Minimizing the Kullback-Leibler distance $KL(P_n, P_\theta)$ upon θ yields

$$\begin{aligned}\theta_{KL} &= \arg \min_{\theta} KL(P_n, P_\theta) \\ &= \arg \min_{\theta} - \sum_{j=1}^k \frac{n_j}{n} \log P_\theta(j) \\ &= \arg \max_{\theta} \sum_{j=1}^k \frac{n_j}{n} \log P_\theta(j) \\ &= \theta_{ML}.\end{aligned}$$

Introduce the *conjugate divergence function* $\tilde{\varphi}$ of φ , inducing the modified Kullback-Leibler, or so-called Likelihood divergence pseudodistance KL_m which therefore satisfies

$$KL_m(P_\theta, P_n) = KL(P_n, P_\theta).$$

We have proved that minimizing the Kullback-Leibler divergence $KL(P_n, P_\theta)$ amounts to minimizing the Likelihood divergence $KL_m(P_\theta, P_n)$ and produces the ML estimate of θ_T .

Kullback-Leibler divergence as defined above by $KL(P_n, P_\theta)$ is related to the way P_n keeps away from P_θ when θ is not equal to the true value of the parameter θ_T generating the observations X_i 's and is closely related with the type of sampling of the X_i 's. In the present case i.i.d. sampling of the $X_{i,\theta}$'s under P_θ results in the asymptotic property, named Large Deviation Sanov property

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_\theta(P_{n,\theta} = P_n | P_n) = -KL(P_{\theta_T}, P_\theta). \quad (2.6)$$

This result can easily be obtained from (2.3) using Stirling formula to handle the factorial terms and the law of large numbers which states that for all j 's, n_j/n tends to $P_{\theta_T}(j)$ as n tends to infinity. Comparing with (2.4) we note that the ML estimator θ_{ML} estimates the minimizer of the natural estimator of $KL(P_{\theta_T}, P_\theta)$ in θ , substituting the unknown measure generating the X_i 's by its empirical counterpart P_n . Alternatively as will be used in the sequel, θ_{ML} minimizes upon θ the Likelihood divergence $KL_m(P_\theta, P_{\theta_T})$ between P_θ and P_{θ_T} substituting the unknown measure P_{θ_T} generating the X_i 's by its empirical counterpart P_n . Summarizing we have obtained:

The ML estimate can be obtained from a LDP statement as given in (2.6), optimizing in θ in the estimator of the LDP rate where the plug-in method of the empirical measure of the data is used instead of the unknown measure P_{θ_T} . Alternatively it holds

$$\theta_{ML} := \arg \min_{\theta} \widehat{KL}_m(P_\theta, P_{\theta_T}) \quad (2.7)$$

with

$$\widehat{KL}_m(P_\theta, P_{\theta_T}) := KL_m(P_\theta, P_n).$$

In the rest of this section we will develop a similar approach for a model \mathcal{P}_Θ whose all members P_θ share the same infinite (countable or not) support S .

The statistical properties of θ_{ML} are obtained under the i.i.d. sampling having generated the observed values.

This principle will be kept throughout this paper: the estimator is defined as maximizing the probability that the simulated empirical measure be close to the empirical measure as observed on the sample, conditionally on it, following the same sampling scheme. This yields a maximum likelihood estimator, and its properties are then obtained when randomness is introduced as resulting from the sampling scheme.

2.2 Maximum likelihood under general distributions and simple sampling

When the support of the generic r.v. X_1 is not finite some of the arguments above are not valid any longer and some discretization scheme is required in order to get occupation probabilities in the spirit of (2.3) or (2.6). Since all distributions P_θ in \mathcal{P}_Θ have infinite support, i.i.d. sampling under any P_θ yields $(X_{1,\theta}, \dots, X_{n,\theta})$ such that

$$P_\theta(P_{n,\theta} = P_n | P_n) = 0$$

for all n , so that we are lead to consider the optimization upon θ of probabilities of the type $P_\theta(P_{n,\theta} \in V(P_n) | P_n)$ where $V(P_n)$ is a (small) neighborhood of P_n . Considering the distribution of the outcomes of the simulating scheme P_θ results in the definition of neighborhoods through partitions of S , hence through the τ_0 -topology.

When P_n is the empirical measure for some observed r.v's X_1, \dots, X_n , an ε -neighborhood of P_n contains distributions whose support is not necessarily finite, and may indeed be equivalent to the measures in the model \mathcal{P}_Θ when defined on the Borel σ -field $\mathcal{B}(S)$.

Let $\mathcal{P}_k := (A_1, \dots, A_k)$ be some partition in \mathfrak{P}_k . Denote

$$V_{k,\varepsilon}(P_n) := \left\{ Q \in \mathcal{M} \text{ such that } \max_{i=1,\dots,k} |P_n(A_i) - Q(A_i)| < \varepsilon \text{ and } Q(A_i) = 0 \text{ if } P_n(A_i) = 0 \right\} \quad (2.8)$$

an open neighborhood of P_n .

We also would define the Kullback-Leibler divergence between two probability measures Q and P on the partition \mathcal{P}_k through

$$KL_{A_k}(Q, P) := \sum_{A_j \in \mathcal{P}_k} \log \left(\frac{Q(A_j)}{P(A_j)} \right) Q(A_j).$$

Also we define the corresponding Likelihood divergence on \mathcal{P}_k through

$$(KL_m)_{\mathcal{P}_k}(Q, P) := KL_{\mathcal{P}_k}(P, Q). \quad (2.9)$$

As in the finite case for any θ in Θ denote $(X_{1,\theta}, \dots, X_{n,\theta})$ a set of n i.i.d. random variables with common distribution P_θ . We have

Lemma 2.1. *For large n*

$$\begin{aligned} \frac{1}{n} \log P_\theta (P_{n,\theta} \in V_{k,\varepsilon}(P_n) | P_n) &\geq -KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta) - \frac{k \log(n+1)}{n} \\ &:= - \inf_{Q \in V_{k,\varepsilon}(P_n)} KL_{\mathcal{P}_k}(Q, P_\theta) - \frac{k \log(n+1)}{n} \end{aligned}$$

Proof. The proof uses similar arguments as in [5] Lemma 4.1. For fixed k and large n , P_{θ_T} belongs to $V_{k,\varepsilon}(P_n)$, by the law of large numbers. Indeed for large n , $P_n(A_j)$ is positive and $|P_{\theta_T}(A_j) - P_n(A_j)| < \varepsilon$ for all j in $\{1, \dots, k\}$. Assuming that for all θ in Θ

$$KL(P_{\theta_T}, P_\theta) < \infty$$

and taking into account the fact (see [11]) that for any probability measures P and Q , $K(P, Q) = \sup_k \sup_{\mathfrak{P}_k} KL_{\mathcal{P}_k}(P, Q)$ where \mathfrak{P}_k is the class of all partitions of S in k sets in $\mathcal{B}(S)$, it follows that

$$KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta) \text{ is finite}$$

for all fixed k and large n . For positive δ let $P^{(n)}$ in $V_{k,\varepsilon}(P_n)$ with

$$KL_{\mathcal{P}_k}(P^{(n)}, P_\theta) < KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta) + \delta.$$

Let $0 < \varepsilon' < \varepsilon$ and non negative numbers r_j , $1 \leq j \leq k$ such that

$$|r_j - P^{(n)}(A_j)| < \varepsilon', \text{ and } r_j = 0 \text{ if } P^{(n)}(A_j) = 0 \text{ and } \sum_{j=1}^k r_j = 1.$$

The probability vector (r_1, \dots, r_k) defines a probability measure R on (S, \mathcal{P}_k) , and R belongs to $V_{k,\varepsilon}(P_n)$. By continuity of the mapping $x \rightarrow x \log \frac{x}{P_\theta(A_j)}$ it is possible to fit the r_j 's such that for all j between 1 and k

$$\left| r_j \log \frac{r_j}{P_\theta(A_j)} - P^{(n)}(A_j) \log \frac{P^{(n)}(A_j)}{P_\theta(A_j)} \right| < \frac{\delta}{k}. \quad (2.10)$$

Indeed since all the P_θ 's share the same support, if $P_\theta(A_j) = 0$ then $P_{\theta_T}(A_j) = 0$ which in turn yields $P_n(A_j) = 0$ which through (2.8) implies $P^{(n)}(A_j) = 0$. This plus the conventions $0/0 = 0$ and $0 \log 0 = 0$ implies that (2.10) holds true for some choice of the r_j 's. Choose further the r_j 's in such a way that $l_j := nr_j$ is an integer for all j . Let

$P_{n,\theta}$ denote the empirical distribution of the $X_{i,\theta}$'s. We now proceed to the evaluation of $P_\theta(P_{n,\theta} \in V_{k,\varepsilon}(P_n) | P_n)$. It holds

$$\begin{aligned} P_\theta(P_{n,\theta} \in V_{k,\varepsilon}(P_n) | P_n) &\geq P_\theta(P_{n,\theta}(A_j) = r_j, 1 \leq j \leq k | P_n) \\ &= \frac{\prod_{j=1}^k l_j!}{n!} \prod_{j=1}^k P_\theta(A_j)^{l_j} \\ &\geq (n+1)^{-k} \exp -n \sum_{j=1}^k r_j \log \frac{r_j}{P_\theta(A_j)} \end{aligned}$$

where we used the same argument as in [5], Lemma 4.1. In turn using (2.10)

$$\begin{aligned} \sum_{j=1}^k r_j \log \frac{r_j}{P_\theta(A_j)} &\leq \sum_{j=1}^k P^{(n)}(A_j) \log \frac{P^{(n)}(A_j)}{P_\theta(A_j)} + \delta \\ &\leq KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta) + 2\delta \end{aligned}$$

and the proof is completed. \square

The reverse inequality is as in [5] p 790: The set $V_{k,\varepsilon}(P_n)$ is completely convex, in the terminology of [5], whence it follows

Lemma 2.2. *For all n*

$$\frac{1}{n} \log P_\theta(P_{n,\theta} \in V_{k,\varepsilon}(P_n) | P_n) \leq -KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta)$$

Lemmas 2.1 and 2.2 link the Maximum Likelihood Principle with the Large deviation statements. Define

$$\theta_{ML} := \arg \max_{\theta} \frac{1}{n} \log P_\theta(P_{n,\theta} \in V_{k,\varepsilon}(P_n) | P_n) \quad (2.11)$$

and

$$\theta_{LDP} := \arg \min_{\theta} -KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta)$$

assuming those parameters defined, possibly not in a unique way. Denote

$$L_{k,\varepsilon}(\theta) := \frac{1}{n} \log P_\theta(P_{n,\theta} \in V_{k,\varepsilon}(P_n) | P_n)$$

and

$$K_{k,\varepsilon}(\theta) := -KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta).$$

We then deduce that

$$\begin{aligned} -\frac{k}{n} \log(n+1) &\leq L_{k,\varepsilon}(\theta_{ML}) - K_{k,\varepsilon}(\theta_{ML}) \leq 0 \\ 0 &\leq -L_{k,\varepsilon}(\theta_{LDP}) - K_{k,\varepsilon}(\theta_{LDP}) \leq \frac{k}{n} \log(n+1) \end{aligned}$$

whence

$$0 \leq L_{k,\varepsilon}(\theta_{ML}) - L_{k,\varepsilon}(\theta_{LDP}) \leq \frac{k}{n} \log(n+1) \quad (2.12)$$

from which θ_{LDP} is a good substitute for θ_{ML} for fixed k and ε in the partitioned based model. Note that the bounds in (2.12) do not depend on the peculiar choice of \mathcal{P}_k in \mathfrak{P}_k .

Fix $k = k_n$ such that $\lim_{n \rightarrow \infty} k_n = \infty$ together with $\lim_{n \rightarrow \infty} k_n/n = 0$. Define the partition \mathcal{P}_k such that $P_n(A_j) = k_n/n$ for all $j = 1, \dots, k$. Hence A_j contains only k sample points. Let $\varepsilon > 0$ such that $\max_{1 \leq j \leq k} |P_{\theta_T}(A_j) - k_n/n| < \varepsilon$. Then clearly P_{θ_T} belongs to $V_{k,\varepsilon}(P_n)$ and $V_{n,\varepsilon}(P_n)$ is included in $V_{k,2\varepsilon}(P_{\theta_T})$. Therefore for any θ it holds

$$KL_{\mathcal{P}_k}(V_{k,2\varepsilon}(P_{\theta_T}), P_\theta) \leq KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta) \leq KL_{\mathcal{P}_k}(P_{\theta_T}, P_\theta) \quad (2.13)$$

which proves that $\inf_\theta KL_{\mathcal{P}_k}(V_{k,\varepsilon}(P_n), P_\theta) = 0$ with attainment on θ' such that $P_{\theta'}$ and P_{θ_T} coincide on \mathcal{P}_k .

We now turn to the study of the RHS term in (2.13). Introducing the likelihood divergence $\tilde{\varphi}$ defined in (2.9) leads

$$KL_{\mathcal{P}_k}(P_{\theta_T}, P_\theta) = (KL_m)_{\mathcal{P}_k}(P_\theta, P_{\theta_T})$$

whence minimizing $KL_{\mathcal{P}_k}(P_{\theta_T}, P_\theta)$ over θ in Θ amounts to minimizing the likelihood divergence $\theta \rightarrow (KL_m)_{\mathcal{P}_k}(P_\theta, P_{\theta_T})$. Set therefore

$$\theta_{LDP, \mathcal{P}_k} := \arg \min_{\theta} KL_{\mathcal{P}_k}(P_{\theta_T}, P_\theta) = \arg \min_{\theta} (KL_m)_{\mathcal{P}_k}(P_\theta, P_{\theta_T}).$$

Based on the σ -field generated by \mathcal{P}_k on S the dual form (1.3) of the Likelihood divergence pseudodistance $(KL_m)_{\mathcal{P}_k}(P_\theta, P_{\theta_T})$ yields

$$\begin{aligned} \arg \min_{\theta} (KL_m)_{\mathcal{P}_k}(P_\theta, P_{\theta_T}) &= \arg \min_{\theta} \sup_{\eta} \sum_{B_j \in \mathcal{P}_k} \tilde{\varphi}\left(\frac{P_\theta}{P_\eta}(A_j)\right) P_\theta(A_j) \\ &\quad - \sum_{B_j \in \mathcal{P}_k} (\tilde{\varphi})^*\left(\frac{P_\theta}{P_\eta}(A_j)\right) P_{\theta_T}(A_j). \end{aligned} \quad (2.14)$$

with $\tilde{\varphi}(x) = -\log x + x - 1$ and $(\tilde{\varphi})^*(x) = -\log(1-x)$. With the present choice for $\tilde{\varphi}$ the terms in P_η vanish in the above expression ; however we complete a full developement,

as required in more involved sampling schemes. Now an estimate of θ_T is obtained substituting P_{θ_T} by P_n in (2.14) leading, denoting n_j the number of X_i 's in A_j

$$\hat{\theta}_{LDP, \mathcal{P}_k} := \arg \min_{\theta} \sup_{\eta} \sum_{A_j \in \mathcal{P}_k} \tilde{\varphi} \left(\frac{P_{\theta}}{P_{\eta}} (A_j) \right) P_{\theta} (A_j) - \sum_{A_j \in \mathcal{P}_k} \frac{n_j}{n} (\tilde{\varphi})^* \left(\frac{P_{\theta}}{P_{\eta}} (A_j) \right).$$

Letting n tend to infinity yields (recall that $k = k_n$)

$$\lim_{n \rightarrow \infty} \sup_{\eta} \left| \left[\sum_{A_j \in \mathcal{P}_k} \tilde{\varphi} \left(\frac{P_{\theta}}{P_{\eta}} (A_j) \right) - \sum_{A_j \in \mathcal{P}_k} (\tilde{\varphi})^* \left(\frac{P_{\theta}}{P_{\eta}} (A_j) \right) P_{\theta_T} (A_j) \right] - \left[\int \tilde{\varphi} \left(\frac{p_{\theta}}{p_{\eta}} (x) \right) p_{\theta} (x) dx - \int (\tilde{\varphi})^* \left(\frac{p_{\theta}}{p_{\eta}} (x) \right) dP_n(x) \right] \right| = 0$$

w.p. 1 which in turn implies

$$\lim_{n \rightarrow \infty} \hat{\theta}_{LDP, \mathcal{P}_k} - \hat{\theta}_{ML} = 0$$

where $\hat{\theta}_{ML}$ is readily seen to be the usual ML estimator of θ defined through

$$\hat{\theta}_{ML} := \arg \sup_{\theta} \prod_{i=1}^n p_{\theta} (X_i).$$

3 Weighted sampling

This section extends the previous arguments for weighted sampling schemes. We will show that the Maximum Likelihood paradigm as defined above can be extended for these schemes, leading to operational procedures involving the minimization of specific divergence pseudodistances defined in strong relation with the distribution of the weights.

The sampling scheme which we consider is commonly used in connection with the bootstrap and is referred to as the *weighted* or *generalized bootstrap*, sometimes called *wild bootstrap*, first introduced by Newton and Mason [9]. The main simplification which we consider in the present setting lies in the fact that we assume that the weights W_i are i.i.d. while being exchangeable random variables in the generalized bootstrap setting.

Let x_1, \dots, x_n be n independent realizations of n i.i.d. r.v's X_1, \dots, X_n with common distribution P_{θ_T} . It will be assumed that

$$\text{For all } \theta \text{ in } \Theta, E_{\theta} X \text{ and } E_{\theta} X^2 \text{ are finite.} \quad (3.1)$$

This entails that both

$$\frac{1}{n} \sum_{i=1}^n x_i \text{ and } \frac{1}{n} \sum_{i=1}^n x_i^2$$

converge P_{θ_T} -a.e. to $E_{\theta_T} X$ and $E_{\theta_T} X^2$ respectively; also the same holds with θ_T substituted by any θ in Θ when x_1, \dots, x_n is sampled under P_{θ} . This assumption is necessary

when studying the properties of the estimates of θ_T and of $\phi(\theta_T, \theta)$ under some alternative θ .

Consider a collection W_1, \dots, W_n of independent copies of W , whose distribution satisfies the conditions stated in Section 1. The weighted empirical measure P_n^W is defined through

$$P_n^W := \frac{1}{n} \sum_{i=1}^n W_i \delta_{x_i}.$$

This empirical measure need not be a probability measure, since its mass may not equal 1. Also it might not be positive, since the weights may take negative values. The measure P_n^W converges almost surely to P_{θ_T} when the weights W_i 's satisfy the hypotheses stated in Section 1. Indeed general results pertaining to this sampling procedure state that under regularity, functionals of the measure P_n^W are asymptotically distributed as are the same functionals of P_n when the X_i 's are i.i.d. Therefore the weighted sampling procedure mimicks the i.i.d. sampling fluctuation in a two steps procedure: choose n values of x_i such that they asymptotically fit to P_{θ_T} , which means

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{x_i} = P_{\theta_T}$$

deterministically and then play the W_i 's on each of the x_i 's. Then get P_n^W , a proxy to the random empirical measure P_n .

For any θ in Θ consider a similar sampling procedure under the weights W_i' 's which are i.i.d. copies of the W_i 's. Let therefore $x_{1,\theta}, \dots, x_{n,\theta}$ denote n i.i.d. realizations of $X_{1,\theta}, \dots, X_{n,\theta}$ with distribution P_θ yielding the empirical measure

$$P_{n,\theta}^{W'} := \frac{1}{n} \sum_{i=1}^n W_i' \delta_{x_{i,\theta}}$$

the corresponding empirical measure. Note that except for the choice of the generating measure P_θ , $P_{n,\theta}^{W'}$ is obtained in the same way as P_n^W . The ML principle turns out to select the value of θ making $P_{n,\theta}^{W'}$ as close as possible from P_n^W , conditionally upon P_n^W .

The resulting estimates are optimal in many respects, as is the classical ML estimator for regular models in the i.i.d. sampling scheme. The proposal which is presented here also allows to obtain optimal estimators for some non regular models. This approach is in line with [3] who developped a whole range of first order optimal estimation procedures in the case of the i.i.d. sampling, based on divergence minimization.

Using the notations of section 1.1.3, we endow $\mathcal{M}(S)$ with τ_0 -topology rather than the weak topology, and define accordingly the σ -field $\mathcal{B}(\mathcal{M})$ on $\mathcal{M}(S)$. Denote by $\mathcal{M}_1(S)$ the space of probability measure on S , endowed with the τ_0 -topology.

3.1 A Sanov conditional theorem for the weighted empirical measure

The procedure which we are going to develop can be stated as follows.

Similarly as in the simple i.i.d. setting select some (small) neighborhood $V_\epsilon(P_n^W)$ of P_n^W and define the MLE of θ_T as the value of θ which optimizes the probability that the simulated empirical measure $P_{n,\theta}^{W'}$ belongs to $V_\epsilon(P_n^W)$. This requires a conditional Sanov type result, substituting Lemmas 2.1 and 2.2. This result is produced in Theorem 3.1 in Section 3.1. In the same vein as in Lemmas 2.1 and 2.2, maximizing in θ this probability amounts to minimizing a LDP rate between P_θ and $V_\epsilon(P_{\theta_T})$. The rate is in strong relation with the distribution of the W_i 's. Call it $\phi^W(V_\epsilon(P_{\theta_T}), P_\theta) := \inf \{ \phi^W(Q, P_\theta), Q \in V_\epsilon(P_{\theta_T}) \}$.

Since ϵ is small, this rate is of order $\phi^W(P_{\theta_T}, P_\theta)$; this is Corollary 3.1 in Section 3.1. Turn to the original data and estimate $\phi^W(P_{\theta_T}, P_\theta)$ by some plug in method to be stated in Section 3.2. Define the ML estimator of θ_T through the minimization of the proxy of $\phi^W(P_{\theta_T}, P_\theta)$. We will prove that minimum divergence estimators play a key role in this setting.

In order to state our conditional Sanov theorem we put forwards the following lemma, which is in the vein of Theorem 2.2 of Najim [10] which states the Sanov large deviation theorem, where the weights are i.i.d random variables. Trashorras and Wintenberger [14] have investigated the large deviations properties of weighted (bootstrapped) empirical measure with exchangeable weights under appropriate assumptions of the weights. Both papers equip $\mathcal{M}(S)$ with the weak topology.

The lemma's proof is deferred to Section 7.

Lemma 3.1. *Assume that $P_\theta(U) > 0$ for any non-empty open set $U \in S$, and that $\lim_{n \rightarrow \infty} P_n = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{x_i} = P_\theta \in \mathcal{M}_1(S)$, where the convergence holds under τ_0 . Then $P_{n,\theta}^W$ satisfies the LDP in $(\mathcal{M}(S), \mathcal{B}(\mathcal{M}))$ equipped with the τ_0 -topology with the good convex rate function:*

$$\begin{aligned} \phi^W(\zeta, P_\theta) &= \sup_{f \in B(\mathbb{R}^d)} \left\{ \int_{\mathbb{R}^d} f(x) \zeta(dx) - \int_{\mathbb{R}^d} M(f(x)) P_\theta(dx) \right\} \\ &= \begin{cases} \int_{\mathbb{R}^d} M^*\left(\frac{d\zeta}{dP_\theta}\right) dP_\theta, & \text{if } \zeta \text{ is a.c. w.r.t. } P_\theta \\ \infty, & \text{otherwise} \end{cases} \end{aligned}$$

where $M^*(x) = \sup_t tx - M(t)$ for all real x and $M(t)$ is the moment generating function of W .

Let $\mathcal{P}_k = (A_1, \dots, A_k)$ denote an arbitrary partition of S with A_i in $B(S)$ for all $i = 1, \dots, k$, and define the pseudometric $d_{\mathcal{P}_k}$ on $\mathcal{M}(S)$ by

$$d_{\mathcal{P}_k}(Q, R) = \max_{1 \leq j \leq k} |Q(B_j) - R(B_j)|, \quad Q, R \in \mathcal{M}(S).$$

For any positive ϵ , let

$$V_\epsilon(P_n^W) = \{Q \in \mathcal{M}(S) : d_{\mathcal{P}_k}(Q, P_n^W) < \epsilon\}$$

denote an open neighborhood of the weighted empirical measure P_n^W in the τ_0 -topology. Then we have the following conditional LDP theorem.

Theorem 3.1. *With the above notation and assuming that P_{θ_T} is absolutely continuous with respect to P_θ , for any positive ϵ , the following conditional LDP result holds*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log P_\theta \left(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n \right) = -\phi^W(V_\epsilon(P_{\theta_T}), P_\theta).$$

Proof. In the following proof, \mathcal{P}_k is an arbitrary partition on S .

$$\begin{aligned} P_\theta \left(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n \right) &= P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_n^W) < \epsilon | P_n \right) \\ &\geq P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_{\theta_T}) + d_{\mathcal{P}_k}(P_{\theta_T}, P_n^W) < \epsilon | P_n \right) \\ &= P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_{\theta_T}) < \epsilon - d_{\mathcal{P}_k}(P_{\theta_T}, P_n^W) | P_n \right). \end{aligned}$$

Since $d_{\mathcal{P}_k}(P_{\theta_T}, P_n^W) \rightarrow 0$ when $n \rightarrow \infty$, for any positive δ and sufficiently large n we have:

$$P_\theta \left(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n \right) \geq P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_{\theta_T}) < \epsilon - \delta \right) = P_\theta \left(P_{n,\theta}^{W'} \in V_{\epsilon-\delta}(P_{\theta_T}) \right).$$

By Lemma 3.1, we obtain the conditioned LDP lower bound

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log P_\theta \left(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n \right) \geq -\phi^W(V_{\epsilon-\delta}(P_{\theta_T}), P_\theta),$$

In a similar way, we obtain the large deviation upper bound

$$\begin{aligned} P_\theta \left(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n \right) &= P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_n^W) < \epsilon | P_n \right) \\ &\leq P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_{\theta_T}) - d_{\mathcal{P}_k}(P_{\theta_T}, P_n^W) < \epsilon | P_n \right) \\ &\leq P_\theta \left(d_{\mathcal{P}_k}(P_{n,\theta}^{W'}, P_{\theta_T}) < \epsilon + \delta' \right) = P_\theta \left(P_{n,\theta}^{W'} \in V_{\epsilon+\delta'}(P_{\theta_T}) \right), \end{aligned}$$

for some positive δ' . We thus obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log P_\theta \left(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n \right) \leq -\phi^W(V_{\epsilon+\delta'}(P_{\theta_T}), P_\theta).$$

Let $\delta'' = \max(\delta, \delta')$, we have

$$\begin{aligned} -\phi^W(V_{\epsilon-\delta''}(P_{\theta_T}), P_{\theta}) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_{\theta} \left(P_{n,\theta}^{W'} \in V_{\epsilon}(P_n^W) | P_n \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{\theta} \left(P_{n,\theta}^{W'} \in V_{\epsilon}(P_n^W) | P_n \right) \leq -\phi^W(V_{\epsilon+\delta''}(P_{\theta_T}), P_{\theta}). \end{aligned}$$

Denote $cl_{\tau_0}(V_{\epsilon}(P_{\theta_T}))$ the closure of the open set $V_{\epsilon}(P_{\theta_T})$ in the τ_0 -topology, and note δ'' is arbitrarily small, then it holds

$$\begin{aligned} -\phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta}) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n} \log P_{\theta} \left(P_{n,\theta}^{W'} \in V_{\epsilon}(P_n^W) | P_n \right) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log P_{\theta} \left(P_{n,\theta}^{W'} \in V_{\epsilon}(P_n^W) | P_n \right) \leq -\phi^W(cl_{\tau_0}(V_{\epsilon}(P_{\theta_T})), P_{\theta}). \end{aligned}$$

It remains to show that

$$\phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta}) = \phi^W(cl_{\tau_0}(V_{\epsilon}(P_{\theta_T})), P_{\theta}). \quad (3.2)$$

Since P_{θ_T} is absolutely continuous with respect to P_{θ} , by Lemma 3.1 we have

$$\phi^W(cl_{\tau_0}(V_{\epsilon}(P_{\theta_T})), P_{\theta}) \leq \phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta}) \leq \phi^W(P_{\theta_T}, P_{\theta}) < \infty. \quad (3.3)$$

Given some small positive constant ω , then there exists $\mu \in cl_{\tau_0}(V_{\epsilon}(P_{\theta_T}))$ satisfying

$$\phi^W(\mu, P_{\theta}) < \phi^W(cl_{\tau_0}(V_{\epsilon}(P_{\theta_T})), P_{\theta}) + \omega.$$

Set $v \in V_{\epsilon}(P_{\theta_T})$, and define $z(\alpha) = \alpha\mu + (1-\alpha)v$, where $0 < \alpha < 1$. Obviously, we have $z(\alpha) \in V_{\epsilon}(P_{\theta_T})$. By Lemma 3.1, the map $\zeta \rightarrow \phi(\zeta, P_{\theta})$ is convex, hence we get

$$\begin{aligned} \phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta}) &\leq \lim_{\alpha \rightarrow 1} \phi^W(z(\alpha), P_{\theta}) \leq \lim_{\alpha \rightarrow 1} \left(\alpha \phi^W(\mu, P_{\theta}) + (1-\alpha) \phi^W(v, P_{\theta}) \right) \\ &= \phi^W(\mu, P_{\theta}) < \phi^W(cl_{\tau_0}(V_{\epsilon}(P_{\theta_T})), P_{\theta}) + \omega, \end{aligned} \quad (3.4)$$

where the equality holds since $\phi^W(v, P_{\theta})$ is finite by (3.3). Combine (3.3) with (3.4) to get (3.2). This proves the conditional large deviation result. \square

Using the above theorem, we obtain the following corollary.

Corollary 3.1. *Under the assumptions of Theorem 3.1, it holds*

$$\lim_{\epsilon \rightarrow 0} \phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta}) = \phi(P_{\theta_T}, P_{\theta}).$$

Proof. By Lemma 3.1, the rate function $\phi^W(\mu, P_{\theta})$ is a good rate function, hence it is lower semi-continuous; this implies

$$\lim_{\epsilon \rightarrow 0} \phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta}) \geq \phi(P_{\theta_T}, P_{\theta}). \quad (3.5)$$

For any $\epsilon > 0$, we have $\phi^W(P_{\theta_T}, P_{\theta}) \geq \phi^W(V_{\epsilon}(P_{\theta_T}), P_{\theta})$; this together with (3.5) completes the proof. \square

3.2 Divergences associated to the weighted sampling scheme

For any Q in $V_\epsilon(P_{\theta_T})$ rewrite the good rate function using the divergence notation

$$\phi^W(Q, P_\theta) = \int M^* \left(\frac{dQ}{dP_\theta} \right) dP_\theta = \int \varphi^W \left(\frac{dQ}{dP_\theta} \right) dP_\theta \quad (3.6)$$

from which $\phi^W(Q, P_\theta)$ is the divergence associated with the divergence function $\varphi^W := M^*$.

Commuting P_{θ_T} and P_θ in (3.6) and introducing the conjugate divergence function $\widetilde{\varphi^W}$ yields

$$\phi^W(Q, P_\theta) = \int \varphi^W \left(\frac{dQ}{dP_\theta} \right) dP_\theta = \int \widetilde{\varphi^W} \left(\frac{dP_\theta}{dQ} \right) dQ = \widetilde{\phi^W}(P_\theta, Q). \quad (3.7)$$

By Theorem 3.1, maximizing $P_\theta(P_{n,\theta}^{W'} \in V_\epsilon(P_n^W) | P_n)$ amounts to minimize $\phi^W(V_\epsilon(P_{\theta_T}), P_\theta)$. A final approximation now yields the form of the criterion to be estimated in order to define the MLE in the present setting. As $\epsilon \rightarrow 0$ the asymptotic order of $\phi^W(V_\epsilon(P_{\theta_T}), P_\theta)$ is equal to $\widetilde{\phi^W}(P_\theta, P_{\theta_T})$ by Corollary 3.1 and (3.7), which is a proxy of $\phi^W(P_{\theta_T}, P_\theta)$ and therefore the theoretical criterion to be optimized in θ .

We now state the dual form of the theoretical criterion $\widetilde{\phi^W}(P_\theta, P_{\theta_T})$ using the dual form (1.3) and (1.4). It holds

$$\widetilde{\phi^W}(P_\theta, P_{\theta_T}) = \sup_{\alpha \in \mathcal{U}} \int \widetilde{h}(\theta, \alpha, x) dP_{\theta_T}(x) \quad (3.8)$$

with

$$\widetilde{h}(\theta, \alpha, x) = \int \left(\widetilde{\varphi^W} \right)' \left(\frac{dP_\theta}{dP_\alpha} \right) dP_\theta - \left(\widetilde{\varphi^W} \right)^\# \left(\frac{dP_\theta}{dP_\alpha}(x) \right)$$

We now turn to the definition of the MLE in this context, estimating the criterion and deriving the estimate.

3.3 MLE under weighted sampling

Using the dual representation of divergences, the natural estimator of $\phi(P_\theta, P_{\theta_T})$ is

$$\widetilde{\phi}_n(P_\theta, P_{\theta_T}) := \sup_{\alpha \in \mathcal{U}} \left\{ \int \widetilde{h}(\theta, \alpha, x) dP_n^W(x) \right\}. \quad (3.9)$$

From now on, we will use $\phi(\theta, \theta_T)$ to denote $\phi(P_\theta, P_{\theta_T})$; whence the resulting estimator of $\phi(\theta_T, \theta_T)$ is

$$\widetilde{\phi}_n(\theta_T, \theta_T) := \inf_{\theta \in \Theta} \widetilde{\phi}_n(\theta, \theta_T) = \inf_{\theta \in \Theta} \sup_{\alpha \in \mathcal{U}} \left\{ \int \widetilde{h}(\theta, \alpha, x) dP_n^W(x) \right\}$$

and the resulting MLE of θ_T is obtained as the minimum dual $\widetilde{\phi}^W$ estimator

$$\widehat{\theta}_{ML,W} := \arg \inf_{\theta \in \Theta} \sup_{\alpha \in \mathcal{U}} \left\{ \int \widetilde{h}(\theta, \alpha, x) dP_n^W(x) \right\}. \quad (3.10)$$

Formula (3.10) indeed defines a Maximum Likelihood estimator, in the vein of (2.1) and (2.11). This estimator requires no grouping nor smoothing.

4 Bahadur slope of minimum divergence tests for weighted data

Consider the test of some null hypothesis $H_0: \theta_T = \theta$ versus a simple hypothesis $H_1: \theta_T = \theta'$.

We consider two competitive statistics for this problem. The first one is based on the estimate of $\widetilde{\phi}^W(P_\alpha, P_\beta)$ defined for all (α, β) in $\Theta \times \Theta$ through

$$T_n(\alpha) := \sup_{\eta \in \Theta} \int \widetilde{\varphi}^W \left(\frac{p_\alpha}{p_\eta} \right) p_\eta d\mu - \int (\widetilde{\varphi}^W)^* \left(\frac{p_\alpha}{p_\beta} \right) dP_n^W$$

where the i.i.d. sample X_1, \dots, X_n has distribution P_β . The test statistics $T_n(\theta)$ converges to 0 under H_0 .

A competitive statistics $\widehat{\psi}(\theta)$ writes

$$\widehat{\psi}(\theta) := \psi(\theta, P_n^W)$$

where $Q \rightarrow \psi(\theta, Q)$ is assumed to satisfy $\psi(\theta, P_\theta) = 0$, and is τ -continuous with respect to Q , which implies that under H_0 the following Large Deviation Principle holds

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \log P_\theta \left(\widehat{\psi}(\theta) \geq t \right) &= -I(t) \\ &= -\inf \left\{ \phi^W(P_\theta, Q), \psi(\theta, Q) \geq t \right\} \end{aligned} \quad (4.1)$$

for any positive t . Also we assume that under $H_1, \widehat{\psi}(\theta)$ converges to $\psi(\theta, P_{\theta'})$

$$\lim_{n \rightarrow \infty} \widehat{\psi}(\theta) =_{\theta'} \psi(\theta, P_{\theta'}) \quad (4.2)$$

where (4.2) stands in probability under θ' .

We now state the Bahadur slope of the test $\widehat{\phi}^W(\theta, \theta)$.

Under H_0

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{2}{n} \log P_\theta (T_n(\theta) \geq t) &= -2 \inf \left\{ \phi^W(P_\theta, Q), \widetilde{\phi}^W(Q, P_\theta) \geq t \right\} \\ &= -2 \inf \left\{ \phi^W(P_\theta, Q), \phi^W(P_\theta, Q) \geq t \right\} \\ &= -2t \end{aligned}$$

while, under H1

$$\lim_{n \rightarrow \infty} T_n(\theta) = \phi^W(P_\theta, P_{\theta'}) \text{ in probability}$$

since P_n^W converges weakly to $P_{\theta'}$.

It follows that the Bahadur slope of the minimum divergence test $\widehat{\phi}^W(\theta, \theta)$ is

$$e_{T_n(\theta)} = -2\phi^W(P_\theta, P_{\theta'}).$$

Let us evaluate the Bahadur slope of the test $\widehat{\psi}(\theta)$.

Following (4.1) and (4.2) it holds

$$e_{\widehat{\psi}(\theta)} = -2 \inf \{ \phi^W(P_\theta, Q), \psi(\theta, Q) \geq \psi(\theta, P_{\theta'}) \}.$$

Since $\inf \{ \phi^W(P_\theta, Q), \psi(\theta, Q) \geq \psi(\theta, P_{\theta'}) \} \leq \phi^W(P_\theta, P_{\theta'})$ it follows that $e_{\widehat{\psi}(\theta)} \leq e_{T_n(\theta)}$.

We have proved

Proposition 4.1. *Under the weighted sampling the test statistics $\widehat{\psi}(\theta)$ is Bahadur efficient among all tests which are empirical versions of τ_0 -continuous functionals.*

5 Weighted sampling in exponential families

In this short section we show that MLE's associated with weighted sampling are specific with respect to the weighting; this is in contrast with the unweighted sampling (i.i.d. simple sampling), under which all minimum divergence estimators coincide with the standard MLE; see [1].

Let

$$p_\theta(x) = \exp[\theta t(x) - C(\theta)] d\mu(x) \tag{5.1}$$

be an exponential family with natural parameter θ in an open set Θ in \mathbb{R}^d , and where μ denotes a common dominating measure for the model. We assume that this family is full i.e. that the Hessian matrix $(\partial^2/\partial\theta^2)C(\theta)$ is definite positive. Recall that under the standard i.i.d. X_1, \dots, X_n sampling the MLE θ_{ML} of θ satisfies

$$\nabla C(\theta)_{\theta_{ML}} = \frac{1}{n} \sum_{i=1}^n t(X_i).$$

Under the weighted sampling W_1, \dots, W_n corresponding to the divergence function φ^W , conditionally on the observed data x_1, \dots, x_n the MLE writes

$$\theta_{ML,W} := \arg \inf_{\theta \in \Theta} \sup_{\alpha \in \mathcal{U}} \int \left(\widetilde{\varphi^W} \right)' \left(\frac{p_\theta}{p_\alpha} \right) p_\theta d\mu - \int \left(\widetilde{\varphi^W} \right)^\# \left(\frac{p_\theta}{p_\alpha} \right) dP_n^W.$$

We prove that $\theta_{ML,W}$ satisfies

$$\nabla C(\theta)_{\theta_{ML,W}} = \frac{1}{n} \sum_{i=1}^n W_i t(x_i).$$

Denote

$$M_n(\theta, \alpha) := \int \left(\widetilde{\varphi^W} \right)' \left(\frac{p_\theta}{p_\alpha} \right) p_\theta d\mu - \int \left(\widetilde{\varphi^W} \right)^\# \left(\frac{p_\theta}{p_\alpha} \right) dP_n^W.$$

Clearly, substituting using (5.1) it holds for all θ

$$\inf_{\theta \in \Theta} \sup_{\alpha \in \mathcal{U}} M_n(\theta, \alpha) \geq M_n(\theta, \theta) = 0. \quad (5.2)$$

We prove that $M_n(\theta_{ML,W}, \alpha)$ is maximal for $\alpha = \theta_{ML,W}$ which closes the proof.

Let X_1, \dots, X_n be n i.i.d. random variables with common distribution P_{θ_T} with θ_T in Θ . Introduce

$$M_n(\theta, \alpha) := \int \varphi' \left(\frac{dP_\theta}{dP_\alpha} \right) dP_\theta - \frac{1}{n} \sum_{i=1}^n \varphi^\# \left(\frac{dP_\theta}{dP_\alpha}(X_i) \right)$$

We prove that

$$\alpha = \theta_{ML,W} \text{ is the unique maximizer of } M_n(\theta_{ML,W}, \alpha) \quad (5.3)$$

which yields

$$\inf_{\theta} \sup_{\alpha} M_n(\theta, \alpha) \leq \sup_{\alpha} M_n(\theta_{ML,W}, \alpha) = M_n(\theta_{ML,W}, \theta_{ML,W}) = 0 \quad (5.4)$$

which together with (5.2) completes the proof.

Define

$$\begin{aligned} M_{n,1}(\theta, \alpha) &:= \int \varphi'(\exp A(\theta, \alpha, x)) \exp B(\theta, x) d\lambda(x) \\ M_{n,2}(\theta, \alpha) &:= \frac{1}{n} \sum_{i=1}^n W_i \exp(A(\theta, \alpha, x_i)) \varphi'(\exp A(\theta, \alpha, x_i)) \\ M_{n,3}(\theta, \alpha) &:= \frac{1}{n} \sum_{i=1}^n W_i \varphi(\exp A(\theta, \alpha, x_i)) \end{aligned}$$

with

$$\begin{aligned} A(\theta, \alpha, x) &:= T(x)'(\theta - \alpha) + C(\alpha) - C(\theta) \\ B(\theta, x) &:= T(x)'\theta - C(\theta). \end{aligned}$$

It holds

$$M_n(\theta, \alpha) = M_{n,1}(\theta, \alpha) - M_{n,2}(\theta, \alpha) + M_{n,3}(\theta, \alpha)$$

with

$$\frac{\partial}{\partial \alpha} M_{n,1}(\theta, \alpha)_{\alpha=\theta} = -\varphi^{(2)}(1) [\nabla C(\theta) - \nabla C(\alpha)_{\alpha=\theta}] = 0$$

for all θ ,

$$\frac{\partial}{\partial \alpha} M_{n,2}(\theta, \alpha)_{\alpha=\theta_{ML,W}} = \varphi^{(2)}(1) \frac{1}{n} \sum_{i=1}^n W_i \left[-T(x_i) + \nabla C(\alpha)_{\alpha=\theta_{ML,W}} \right] = 0$$

and

$$\frac{\partial}{\partial \alpha} M_{n,3}(\theta_{ML,W}, \alpha) = \frac{1}{n} \sum_{i=1}^n W_i \left[-T(x_i) + \nabla C(\alpha)_{\alpha=\theta_{ML,W}} \right] = 0$$

where the two last displays hold iff $\alpha = \theta_{ML}$. Now

$$\begin{aligned} \frac{\partial^2}{\partial \alpha^2} M_{n,1}(\theta, \alpha)_{\alpha=\theta_{ML,W}} &= (\varphi^{(3)}(1) + 2\varphi^{(2)}(1)) (\partial^2 / \partial \theta^2) C(\theta_{ML,W}) \\ \frac{\partial^2}{\partial \alpha^2} M_{n,2}(\theta, \alpha)_{\alpha=\theta_{ML,W}} &= (\varphi^{(3)}(1) + 4\varphi^{(2)}(1)) (\partial^2 / \partial \theta^2) C(\theta_{ML,W}) \\ \frac{\partial^2}{\partial \alpha^2} M_{n,3}(\theta_{ML}, \alpha)_{\alpha=\theta_{ML,W}} &= \varphi^{(2)}(1) (\partial^2 / \partial \theta^2) C(\theta_{ML,W}), \end{aligned}$$

whence

$$\begin{aligned} \frac{\partial}{\partial \alpha} M_n(\theta, \alpha)_{\alpha=\theta_{ML,W}} &= 0 \\ \frac{\partial^2}{\partial \alpha^2} M_n(\theta, \alpha)_{\alpha=\theta_{ML,W}} &= -\varphi^{(2)}(1) (\partial^2 / \partial \theta^2) C(\theta_{ML,W}) \end{aligned}$$

which proves (5.3), and closes the proof.

In contrast with the i.i.d. sampling case minimum divergence estimators in exponential families under appropriate weighted sampling do not coincide independently upon the divergence.

6 Weak behavior of the weighted sampling MLE's

The distribution of the estimator is obtained under the sampling scheme which determines its form. Hence under the weighted sampling one. So the observed sample x_1, \dots, x_n is considered non random, and is assumed to satisfy

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \delta_{x_i} = P_{\theta_T}$$

and randomness is due to the set of i.i.d. weights W_1, \dots, W_n .

All those estimators can be written as approximate linear functionals of the weighted empirical measure P_n^W . Therefore all the proofs in [3] can be adapted to the present estimators. Even the asymptotic variances of the estimators are the same, and subsequently, Wilk's tests, confidence areas, minimum sample sizes certifying a given asymptotic power, etc, remain unchanged. The only arguments to be noted are the following: All arguments pertaining to laws of large numbers for functionals of the empirical measure carry over to the present setting, conditionally on the observations x_1, \dots, x_n . Indeed consider a statistics

$$U_n := \frac{1}{n} \sum_{i=1}^n W_i f(x_i)$$

where the function f satisfies

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f(x_i) = \mu_{1,f} < \infty$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n f^2(x_i) = \mu_{2,f} < \infty.$$

Then clearly

$$\lim_{n \rightarrow \infty} EU_n = \mu_{1,f}$$

and

$$\lim_{n \rightarrow \infty} Var U_n = \mu_{2,f} - (\mu_{1,f})^2.$$

Weak behavior of the estimates follow also from similar arguments: Consider for example the statistics

$$T_n := \sqrt{n} (U_n - \mu_{1,f}) / \sqrt{\mu_{2,f} - (\mu_{1,f})^2}.$$

Using Lindeberg Central limit theorem for triangular arrays, we obtain that T_n is asymptotically standard normal conditionally upon x_1, \dots, x_n . It follows that the limit distributions of $\widehat{\phi}^W(\theta, \theta_T)$ and of $\widehat{\theta}_{ML,W}$ conditionally on x_1, \dots, x_n coincide with those of $\phi_n(\theta, \theta_T)$ and of $\widehat{\theta}_n$ as stated in [3] under the i.i.d. sampling. Also all results pertaining to tests of hypotheses are similar, as is the possibility to handle non regular models.

7 Proof of Lemma 3.1

Proof. Recall that $B(S)$ denotes the class of all bounded measurable functions on S . Write $B'(S)$ as the algebraic dual of $B(S)$. We equip $B'(S)$ with $B(S)$ -topology, it is the weakest topology which makes continuous the following linear functional:

$$\zeta \mapsto \langle f, \zeta \rangle: B'(S) \rightarrow \mathbb{R}, \text{ for all } f \text{ in } B(S),$$

where $\langle f, \zeta \rangle$ denotes the value of $f(\zeta)$. It follows that $\mathcal{M}(S)$ is included in $B'(S)$ and is endowed with the τ_0 -topology induced by $B(S)$. Construct the projection: $p_{f_1, \dots, f_m}: B'(S) \rightarrow \mathbb{R}^m, m \in \mathbb{Z}_+$, namely, $p_{f_1, \dots, f_m}(\zeta) = (\langle f_1, \zeta \rangle, \dots, \langle f_m, \zeta \rangle), f_1, \dots, f_m \in B(S)$. Then for $p_{f_1, \dots, f_m}(P_{n, \theta}^W) = (\langle f_1, P_{n, \theta}^W \rangle, \dots, \langle f_m, P_{n, \theta}^W \rangle)$ we define the corresponding limit logarithm moment generating function as follows

$$\begin{aligned} h(t) &:= \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(\exp(n \langle t, Y_m \rangle)) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{E}(\exp(\sum_{j=1}^m \langle t_j f_j, \sum_{i=1}^n W_i \delta_{x_i} \rangle)) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \log \mathbb{E} \exp \left(\sum_{j=1}^m t_j f_j(x_i) W_i \right) = \int \left(\sum_{j=1}^m M(t_j f_j) \right) dP_\theta \end{aligned}$$

where $t = (t_1, \dots, t_m) \in \mathbb{R}^m$ and $Y_m = (\langle f_1, P_{n, \theta}^W \rangle, \dots, \langle f_m, P_{n, \theta}^W \rangle)$. The function $h(t)$ is finite since $f \in B(S)$. $M(f)$ is Gateaux-differentiable since the function $s \rightarrow M(f + sg)$ is differentiable at $s = 0$ for any $f, g \in B(S)$

$$\frac{d}{ds} M(f + sg)|_{s=0} = \frac{\int g e^f dP_W}{\int e^f dP_W},$$

where P_W is the law of W . Further, the Gateaux-differentiability of $M(f)$ together with the interchange of integration and differentiation justified by dominated convergence theorem show that $h(t)$ is also Gateaux-differentiable in $t = (t_1, \dots, t_m)$. Hence by the Gartner-Ellis Theorem (see e.g. Theorem 2.3.6 of [6]), $p_{f_1, \dots, f_m}(P_{n, \theta}^W)$ satisfies the LDP in \mathbb{R}^m with the good rate function

$$\begin{aligned} \Phi_{f_1, \dots, f_m}(\langle f_1, \zeta \rangle, \dots, \langle f_m, \zeta \rangle) &= \sup_{t_1, \dots, t_m \in \mathbb{R}} \left\{ \sum_{i=1}^m t_i \langle f_i, \zeta \rangle - \int M \left(\sum_{i=1}^m t_i f_i \right) dP_\theta \right\} \\ &\leq \sup_{f \in B(S)} \Phi_f(\langle f, \zeta \rangle) := \phi^W(\zeta, P_\theta). \end{aligned} \quad (7.1)$$

Since m is arbitrary positive integer, by Dawson-Gartner's Theorem (see e.g. Theorem 4.6.1 of [6]), $P_{n, \theta}^W$ satisfies the LDP in $B'(S)$ with the good rate function $\phi^W(\zeta, P_\theta)$, which

is:

$$\begin{aligned}\phi^W(\zeta, P_\theta) &= \sup_{f \in B(S)} \Phi_f(\langle f, \zeta \rangle) = \sup_{f \in B(S)} \left\{ \int_S f(x) \zeta(dx) - \int_S M(f) P_\theta(dx) \right\} \\ &= \int_S M^* \left(\frac{d\zeta}{dP_\theta} \right) dP_\theta,\end{aligned}$$

note that $B'(S)$ is endowed with the τ_0 -topology, the proof of last equality is given below. Here we always assume ζ is absolutely continuous with respect to P_θ , otherwise $\phi^W(\zeta, P_\theta) = \infty$. Consider $\mathcal{M}(S) \subset B'(S)$, and set $\phi^W(\zeta, P_\theta) = \infty$ when $\zeta \notin \mathcal{M}(S)$. Hence $P_{n,\theta}^W$ satisfies the LDP in $\mathcal{M}(S)$ with the rate function $\phi^W(\zeta, P_\theta)$, for $\zeta \in \mathcal{M}(S)$. As mentioned before, $\mathcal{M}(S)$ is endowed with the topology induced by $B'(S)$, namely the τ_0 -topology. Now we give another representation of the rate function $\phi^W(\zeta, P_\theta)$. We have:

$$\begin{aligned}& \sup_{\zeta \in \mathcal{M}(S)} \left\{ \int_S f(x) \zeta(dx) - \int_S M^* \left(\frac{d\zeta}{dP_\theta} \right) dP_\theta \right\} \\ &= \sup_{\zeta \in \mathcal{M}(S)} \left\{ \int_S \left(\int_S f d\zeta - M^* \left(\frac{d\zeta}{dP_\theta} \right) \right) dP_\theta \right\} \leq \int_S M(f) dP_\theta,\end{aligned}$$

where the inequality holds from the duality lemma and when $d\zeta = (dP_\theta)M'(f)$ the equality holds. Using once again the duality lemma, we obtain the following identity:

$$\int_S M^* \left(\frac{d\zeta}{dP_\theta} \right) dP_\theta = \sup_{\zeta \in \mathcal{M}(S)} \left\{ \int_S f(x) \zeta(dx) - \int_S M(f) dP_\theta \right\} = \phi^W(\zeta, P_\theta).$$

The convexity of the rate function $\zeta \rightarrow \phi^W(\zeta, P_\theta)$ holds from Theorem 7.2.3 of [6] where they show the convexity of $\phi^W(\zeta, P_\theta)$ on $\mathcal{M}(S)$ endowed with $B(S)$ -topology. Hence this is also applied to τ_0 -topology which is induced by $B(S)$ -topology. This completes the proof of the lemma. \square

Remark 7.1. By the classical Gartner-Ellis Theorem, in (7.1), the essential smoothness of $h(t)$ is needed for Φ_{f_1, \dots, f_m} to be a “good rate function”. But on a locally convex Hausdorff topological vector space, the essential smoothness of $h(t)$ can be reduced to Gateaux differentiability; see Corollary 4.6.14 (page 167) and the proof Theorem 6.2.10 (page 265) of [6].

Remark 7.2. Since $\Phi_{f_1, \dots, f_m}(\langle f_1, \zeta \rangle, \dots, \langle f_m, \zeta \rangle)$ is a good rate function in \mathbb{R}^m , its level sets $\Phi_{f_1, \dots, f_m}^{-1}(\alpha) = \{(y_1, \dots, y_m) \in \mathbb{R}^m : \Phi_{f_1, \dots, f_m}(y_1, \dots, y_m) \leq \alpha\}$ are compact, for all α in $[0, \infty)$. Denote the projective limit of $\Phi_{f_1, \dots, f_m}^{-1}(\alpha)$ by $\Phi_f^{-1}(\alpha) = \varprojlim \Phi_{f_1, \dots, f_m}^{-1}(\alpha)$. According to Tychonoff’s theorem, the projective limit $\Phi_f^{-1}(\alpha)$ of the compact set $\Phi_{f_1, \dots, f_m}^{-1}(\alpha)$ is still compact, so $\phi^W(\zeta, P_\theta) = \sup_{f \in B(S)} \Phi_f(\langle f, \zeta \rangle)$ is also a good rate function in $(\mathcal{M}(S), \mathcal{B}(\mathcal{M}))$.

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